assumed that the general term for the error may be expressed in the form

$$\delta(x, y, t) = e^{\alpha t} e^{i\beta x} e^{i\gamma y} \quad (4)$$

or equivalently

$$\delta_{j,k,n} = e^{\alpha n\Delta t} e^{i\beta j\Delta x} e^{i\gamma k\Delta y} \quad (5)$$

If one lets $\xi = e^{\alpha\Delta t}$ and $r = \Delta t/(\Delta x)^2$, Eqs. (5) and (3) may be combined to give

$$\xi^2 - 2A\xi + 1 = 0 \quad (6)$$

where

$$A = 1 - \frac{8Dr^2}{m}\left[\sin^2\frac{\beta\Delta x}{2} + \sin^2\frac{\gamma\Delta y}{2}\right]^2 \quad (7)$$

It can be shown that the error will not grow with increasing time as long as the following necessary and sufficient condition is applied

$$|\xi| \leq 1 \quad (8)$$

In terms of A, this requirement becomes

$$-2 \leq -\frac{8Dr^2}{m}\left[\sin^2\frac{\beta\Delta x}{2} + \sin^2\frac{\gamma\Delta y}{2}\right]^2 \leq 0 \quad (9)$$

The right-hand inequality is satisfied for all values of r, but the left-hand inequality imposes the following restriction on r

$$r \leq \tfrac{1}{4}(m/D)^{1/2} \quad (10)$$

By satisfying this condition, the calculation of the transient response of an impulsively-loaded flat plate, for example, is assured to be stable.

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Nearly Self-Similar Unsteady Motion

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THE self-similar motion associated with a very strong shock in one-dimensional unsteady flow or the analogous, hypersonic small-disturbance flow has been studied by many investigators. A review and list of references can be found in an article by Mirels.[1] As an extension, he also treated the problems of perturbation on self-similar solutions in hypersonic small-disturbance flow.[2] The present study considers unsteady motions that approach self-similarity asymptotically at large time. The strong-shock assumption is made, and the problem is attacked by using asymptotic expansions in large time such that the leading approximation is a self-similar solution. Although the study ultimately aims toward determining the entire asymptotic flow field,† the purpose of this brief note is to emphasize and study the following specific considerations.

1. Nonuniformity of Approximation

The perturbation equations governing the second term of the asymptotic expansions are singular at the face of the piston. In consequence, some of the assumed asymptotic expansions are not valid in this neighborhood; however, several methods may be used to render the approximation uniformly valid. One is the method of matched asymptotic expansions with one set of expansions valid in an "*outer*" region *away* from the piston and another set valid in an "*inner*" region *near* the piston.[3] In Mirels' treatment of nearly self-similar motion, this nonuniformity is not mentioned and his solution corresponds only to the outer expansions of the present formulation.

2. Possible Disparity in Perturbation Powers

Although it seems natural to assume the same power-law perturbation for both the body and shock wave, the possibility of a disparity in the perturbation powers cannot be ruled out. In fact, the recent studies of Guiraud[4] and Messiter[5] of a special power-law perturbation, that is a result of the displacement effect of an entropy layer, to a particular self-similar solution (the blast wave) provide an example of such disparity. (Vaglio-Laurin's different result[6] on the same problem is due to an error made in his matching process.[7]) In Mirels' work, the cases with disparity in the perturbation power are implicitly excluded from consideration.‡

Formulation of the Problem

Consider the motion of the piston to have an asymptotic behavior at large time

$$x_b = t^m(\eta_{b0} + 1/t^N + \ldots) \quad (1)$$

where x and t are nondimensional distance and time, respectively. A constant η_{b0} is to be determined later, and m and N are positive numbers§ with $2/(j + 3) < m < 1$ and $j = 0$, 1, and 2 for plane, cylindrical and spherical motion, respectively. The associated motion of a shock wave can be assumed to have an asymptotic behavior at large time

$$x_s = t^m(1 + a_k/t^{kN} + \ldots) \quad (2)$$

where the constants k and a_k are to be determined. Although intuition indicates $k = 1$, it will be established that cases exist in which $k \neq 1$. The asymptotic expansions for the nondimensional velocity v, pressure p, and density ρ will be assumed according to their form behind the shock wave, which is considered to be very strong. Let

$$\left.\begin{aligned} \eta &= x/t^m \\ v &= mt^{m-1}[V_0(\eta) + V_k(\eta)/t^{kN} + \ldots] \\ p &= m^2t^{2(m-1)}\left[P_0(\eta) + \frac{P_k(\eta)}{t^{kN}} + \ldots\right] \\ \rho &= D_0(\eta) + \frac{D_k(\eta)}{t^{kN}} + \ldots \end{aligned}\right\} \quad (3)$$

Substitution of Eqs. (3) into the conservation equations, the strong shock conditions and the boundary condition $v_b =$

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† The analysis of this problem will be presented more fully in a subsequent paper in collaboration with H. K. Cheng.

‡ The author is informed by H. K. Cheng that recently Mirels independently observed the possibility of disparity in the perturbation powers.

§ The analysis holds for small instead of large time, if N is a negative number.

dx_b/dt leads to the statement of the basic problem

$$\left.\begin{aligned} &(V_0 - \eta)D_0' + D_0V_0' + j\, V_0D_0/\eta = 0 \\ &(V_0 - \eta)V_0' + \frac{P_0'}{D_0} - \frac{(j+1)}{2}\beta V_0 = 0 \\ &(V_0 - \eta)\left(\frac{P_0'}{P_0} - \gamma\frac{D_0'}{D_0}\right) = (j+1)\beta \\ &P_0(1) = V_0(1) = (2/\gamma) + 1 \end{aligned}\right\} \quad (4)$$

$$D_0(1) = (\gamma+1)/(\gamma-1) \qquad V_0(\eta_{b0}) = \eta_{b0}$$

and the perturbation problem

$$\left.\begin{aligned} &\frac{V_k'}{(\eta - V_0)} - \frac{D_k'}{D_0} + \left(\frac{D_0'}{D_0} + \frac{j}{\eta}\right)\frac{V_k}{(\eta - V_0)} + \\ &\qquad \frac{[V_0' + j(V_0/\eta) - k(N/m)]}{(\eta - V_0)}\frac{D_k}{D_0} = 0 \\ &(\eta - V_0)V_k' - \frac{P_k'}{D_0} + \\ &\qquad \left(\frac{j+1}{2}\beta + k\frac{N}{m} - V_0'\right)V_k + \frac{P_0'}{D_0}\frac{D_k}{D_0} = 0 \\ &\frac{P_k'}{P_0} - \gamma\frac{D_k'}{D_0} - \left(\frac{P_0'}{P_0} - \gamma\frac{D_0'}{D_0}\right)\frac{V_k}{(\eta - V_0)} + \\ &\qquad \gamma\left[\frac{D_0'}{D_0} - k\frac{N/m}{(\eta - V_0)}\right]\frac{D_k}{D_0} - \\ &\qquad \left[\gamma\frac{D_0'}{D_0} - \frac{(j+1)\beta + kN/m}{(\eta - V_0)}\right]\frac{P_k}{P_0} = 0 \\ &V_k(1) = a_k\left[\frac{2}{(\gamma+1)}\frac{(m - kN)}{m} - V_0'(1)\right] \\ &P_k(1) = a_k\left[\frac{4}{(\gamma+1)}\frac{(m - kN)}{m} - P_0'(1)\right] \\ &D_k(1) = a_k[-D_0'(1)] \\ &\frac{V_k(\eta_{b0})}{t^{kN}} + \ldots = \frac{1}{t^N}\left[\frac{m - N}{m} - V_0'(\eta_{b0})\right] - \\ &\qquad \frac{1}{t^{2N}}\frac{V_0''(\eta_{b0})}{2} - \frac{1}{t^{3N}}\frac{V_0'''(\eta_{b0})}{6} + \ldots \end{aligned}\right\} \quad (5)$$

where the prime denotes differentiation with respect to η, and

$$\beta \equiv [2(1 - m)]/[(j+1)m] \qquad 0 < \beta < 1$$

Solution of the Basic Problem and Determination of k

The basic problem can be solved by numerical integration starting from the shock wave ($\eta = 1$) and extending to the piston ($\eta = \eta_{b0}$). The unperturbed position of piston η_{b0} is found by satisfying the last of Eqs. (4). The details of integration will not be treated here. Instead, the asymptotic behavior of the basic solutions near the piston will be investigated. They are

$$\left.\begin{aligned} V_0 &= \eta_{b0} + L\theta + j\frac{(1-L)}{2\eta_{b0}}\theta^2 + \\ &\qquad \frac{(j+1)(1-L)}{2(2+\sigma)}\beta\eta_{b0}\frac{C_d}{C_P}\theta^{2+\sigma} + 0(\theta^3) \\ D_0 &= C_d\theta^\sigma\left[1 + j\frac{\sigma}{2\eta_{b0}}\theta + \frac{(j+1)\beta\eta_{b0}C_d}{(2+\sigma)C_P}\right. \\ &\qquad \left.\times\, \theta^{1+\sigma} + 0(\theta^2)\right] \\ P_0 &= C_P\left[1 + \frac{(j+1)\beta\eta_{b0}C_P}{2(1+\sigma)C_P}\theta^{1+\sigma} + 0(\theta^{2+\sigma})\right] \end{aligned}\right\} \quad (6)$$

where the constants C_P and C_d are determined by satisfying the shock conditions of Eqs. (4) and

$$\theta \equiv \eta - \eta_{b0}$$
$$L \equiv 1 - (j+1)[1 - (\beta/\gamma)]$$
$$\sigma \equiv \beta/(\gamma - \beta)$$

Notice that D_0 vanishes at the piston position ($\theta = 0$). The value of σ varies between 0 and 3.

Now with all derivatives of V_0 at the piston ($\eta = \eta_{b0}$) known, the last of Eqs. (5) can be used to determine k, provided that the expansion for velocity in Eqs. (3) is valid at the piston. Obviously, k must be such that the left-hand-side term is of the same order of magnitude as the dominating term on the right-hand side; otherwise, either that equation cannot be satisfied or a trivial solution results. Therefore, k is indeed unity as intuition indicates unless the coefficient of the leading term on the right-hand side vanishes; in that case, k is two. If the coefficients of the first two leading terms vanish, then k could be 3, and so on. With μ defined by

$$\mu \equiv \frac{N}{m(1-L)} = \frac{N}{m(j+1)[1 - (\beta/\gamma)]}$$

it is found that $k = 1$ if $\mu < 1$¶; $k = 2$ if $\mu = 1$; and $j \neq 0$, $k = 3$ if $\mu = 1$, $j = 0$ and $\sigma < 1$. The last case could be $k = 4$ or 5, if the value of σ is between 1 and 2, or between 2 and 3, respectively. The most interesting situation is certainly that when disparity occurs in the perturbation power for the piston and the shock ($\mu = 1$). It can be shown that this case corresponds to perturbation of hypersonic flow over power-law slender bodies due to the displacement effect of the entropy layer.

Solution of the Perturbation Problem and Determination of a_k

The numerical integration of the perturbation problem and the determination of a_k are described by Mirels.[2] He noticed that Eqs. (5) have a singularity at the unperturbed piston position and asymptotic solutions of the perturbation problem near the piston are needed to aid the numerical integration. Actually, due to this singularity nonuniformity in validity arises and Mirels' solution corresponds only to the outer solution in the method of matched asymptotic expansions. Near the piston the perturbation solutions behave as

$$\left.\begin{aligned} &\frac{V_k}{\eta - V_0} = A_k(1 + l_{1k}\theta^{1+\sigma} + \ldots) + \frac{B_k}{\theta}\left(1 - \right. \\ &\qquad \left.\frac{j}{2\eta_{b0}}\theta + m_{1k}\theta^{1+\sigma} + \ldots\right) + \frac{C_k}{\theta^{k\mu}}(n_{1k}\theta^{1+\sigma} + \ldots) \\ &\frac{P_k}{P_0} = A_k\left(\frac{\gamma}{k\mu} + l_{2k}\theta^{1+\sigma} + \ldots\right) + \\ &\qquad \frac{B_k}{\theta}(m_{2k}\theta^{1+\sigma} + \ldots) + \frac{C_k}{\theta_{k\mu}}(n_{2k}\theta^{1+\sigma} + \ldots) \\ &\frac{D_k}{D_0} = A_k\left(\frac{1+\sigma}{k\mu} + l_{3k}\theta^{1+\sigma} + \ldots\right) + \frac{B_k}{\theta}\times \\ &\left(\frac{\sigma}{k\mu - 1} - \frac{\sigma}{k\mu - 1}\cdot\frac{j}{2\eta_{b0}}\theta + m_{3k}\theta^{1+\sigma} + \ldots\right) + \\ &\qquad \frac{C_k}{\theta^{k\mu}}\left(1 - j\frac{k\mu}{2\eta_{b0}}\theta + n_{3k}\theta^{1+\sigma} + \ldots\right) \end{aligned}\right\} \quad (7)$$

where A_k, B_k, and C_k are integration constants, and l_k's, m_k's, and n_k's are constants determined by satisfying the

¶ It will be seen later that for $\mu > 1$ the expansion for velocity in Eqs. (3) is not valid at $\eta = \eta_{b0}$, and the argument used here to determine k does not hold; therefore, these cases are excluded here.

perturbation differential equations. The cases mentioned in the last section will now be discussed.

$\mu < 1, k = 1$

In this case the expansions for velocity and pressure are uniformly valid. The first of Eqs. (7) and the last of Eqs. (5) can be used to show that

$$V_1(\eta_{b0}) = (1 - L)B_1$$

and

$$B_1 = 1 - \mu \qquad (8)$$

Once B_1 is known, A_1, C_1, and a_1 can be determined as in Mirels' work.[2] However, the expansion for density is not uniformly valid; namely, in an inner region where θ is $0(t^{-N})$, the second term in the expansion D_1t^{-N} attains the same order of magnitude as the first term D_0. Using the method of matched asymptotic expansions, one can show that in the inner region

$$\rho = (C_d/t^{\sigma N}) \times \left[(\bar{\theta} - 1)^{\sigma} + C_1 \frac{(\bar{\theta} - 1)^{\sigma - \mu}}{t^{(1-\mu)N}} + \ldots \right] \qquad \bar{\theta} = t^N\theta \quad (9)$$

which is still not valid in the innermost region where $(\bar{\theta} - 1)$ is $0(t \exp\{-[(1-\mu)N/\mu]\})$. In this innermost region the density cannot be determined by the governing equations because it depends on the initial history of the shock; however, given the initial shock motion and using the particle-isentropic condition, the complete asymptotic flow field can be determined. Such dependence of density on the initial history of the shock may happen in the inner region in other cases.

$\mu = 1, k = 1$

Although the asymptotic behavior of V_1 near the piston will be different from the preceding case, with logarithmic terms appearing, it can be proved that the log term is not the dominating term at $\eta = \eta_{b0}$ and, therefore, still $B_1 = 1 - \mu$ which is zero in this case. It follows that this case yields only trivial solutions.

$\mu = 1, k = 2$

The expansion for velocity is still uniformly valid and, as in preceding cases, B_k can be shown to be

$$B_2 = -j/2\eta_{b0} \qquad (10)$$

It is obvious that if $j = 0$ the preceding is a trivial solution. For $j = 1$ or 2, A_2, C_2, and a_2 can again be determined. The outer expansion for density is still not valid in the inner region $\theta = 0(t^{-N})$ where now the density already depends on the initial shock motion. The outer expansion for pressure may or may not be uniformly valid depending on whether σ is larger or smaller than unity.

$\mu = 1, j = 0, \sigma < 1, k = 3$

All the expansions of Eqs. (3) are not valid near the piston, and the last of Eqs. (5) can no longer be used to determine A_k, B_k, C_k, and a_k as before. They can only be determined from matching with the inner solutions.

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Some Experimental Observations on the Nonlinear Vibration of Cylindrical Shells

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Nomenclature

h = shell thickness (in.)
L = unsupported length of shell (in.)
n = number of circumferential waves in shell vibration mode
p_0 = amplitude of Jensen driver output (psi)
R = shell radius (in.)
V = voltage input to Jensen driver (v rms)
δ = driver-to-shell (or pressure transducer) spacing (in.)

Introduction

In a note on the nonlinear vibration of cylindrical shells, Evensen[1] indicated that, contrary to the findings of Chu[2] and Nowinski,[3] the nonlinearity revealed by his preliminary investigations was of the "softening" type, and the vibrations were only slightly nonlinear. In subsequent work, Evensen[4] treated the analogous problem for a thin-walled ring in great detail both theoretically and experimentally. He found that the ring vibrations exhibited only a small "softening" type of nonlinearity. As a consequence of the similarity between the two problems, these findings lend support to the earlier predictions. However, at the present time, there seems to be no quantitative experimental data available to substantiate these predictions for the complete cylindrical shell.

In preparing for flutter experiments,[5] the author performed vibration tests on several cylindrical shells. The results of these tests included some qualitative observations and some quantitative data on large amplitude vibrations, and these are reported herein.

Experimental

The forementioned shells were thin-walled seamless circular cylinders made of copper by an electroplating process. The large amplitude vibration test was carried out on a 0.0044-in. thick shell (radius-to-thickness ratio of 1820) mounted on the flutter model. Motion of the shell skin was measured with an inductance pickup that could be traversed both axially and circumferentially under the shell without touching it. Full details of the shells and flutter model are reported elsewhere.[5]

The shell vibrations were driven by a Jensen model D-40 acoustic driver whose acoustic output was focused through a conical nozzle with a 0.25-in.-diam exit hole. The driver was positioned midway between the ends of the shell with the

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